

Vanishing cycles of smoothable isolated Cohen-Macaulay codimension 2 singularities of type 2

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Abstract

We extend the results from the previous paper by A. Frühbis-Krüger and the author [5] to the vanishing topology of those singularities in the title. Studying the case of possibly non-isolated singularities in the Tjurina-transform, we reveal that in dimension 3 and 2 there always is exactly one special vanishing cycle in degree 2 closely related to the determinantal structure of the singularity.

The results of this paper give a detailed insight in the vanishing topology of a certain class of isolated determinantal singularities: Those given by the maximal minors of 3×2 matrices with analytic entries. It turns out that the homology of the Milnor fiber reflects the determinantal structure, see theorem 1.11. We use the Tjurina modification and its compatibility with deformations as introduced in [5] and combine it with results about the vanishing topology of non-isolated complete intersection singularities. The latter are obtained as generalizations and adaptations of the work by D. Siersma, M. Tibar and Y. Yomdin [16], [17], [18], [10]. While computations of further examples beyond Cohen-Macaulay type 2 yield that the observed phenomena still hold true for determinantal singularities defined by bigger matrices, the methods applied in this paper seem to be rather exhausted.

The article is structured as follows. First we will review known facts for isolated Cohen-Macaulay codimension 2 singularities and the Tjurina modification in section 1. After stating the main theorem of this paper, we give an example to sketch the structure of the proof. In section 2 we develop the necessary results for the local complete intersection line singularities with a view towards their applications for determinantal ones. This will all be put together in section 3, where we compute the homology groups for the deformed Tjurina transform in a “generic rank 1 perturbation” and finally for the Milnor fiber of any Cohen-Macaulay codimension 2 singularity of Cohen-Macaulay type $t = 2$ and dimension 2 or 3.

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1 Known facts and techniques

Definition 1.1. *An isolated Cohen-Macaulay codimension 2 (ICMC2) singularity $(X, 0) \subset (\mathbb{C}^N, 0)$ is, as the name yields, the germ of an analytic Cohen-Macaulay scheme of codimension 2 in \mathbb{C}^N such that any representative X of $(X, 0)$ is smooth in a punctured neighborhood of the origin. The Cohen-Macaulay type of $(X, 0)$ can be defined as*

$$t := (\min \# \text{generators of } I(X)) - 1,$$

where $I(X)$ is the ideal in $\mathbb{C}\{\underline{x}\}$ associated to $(X, 0)$ at 0.

The Hilbert Burch theorem says that being Cohen-Macaulay of codimension 2 is equivalent to the fact that $I = I(X)$ is given by the t -minors of the syzygy-matrix A of I , see [1]. This matrix is always of the form

$$A \in \text{Mat}(t, t+1; \mathbb{C}\{\underline{x}\}).$$

Furthermore Schaps [15] showed that any deformation of $(X, 0)$, i.e. an embedding of $(X, 0)$ in a flat family, is described as a perturbation of this matrix A . This means the minors of the perturbed matrix, whose entries now also depend on the deformation parameters, define the total space of the deformation.

In particular this means that every Cohen-Macaulay singularity is a *determinantal singularity* of type $(t, t+1, t)$ in the following sense.

Definition 1.2. *A germ $(X, 0) \subset (\mathbb{C}^N, 0)$ is called a determinantal singularity of type (m, n, s) , if there exists a holomorphic map germ*

$$A : (\mathbb{C}^N, 0) \rightarrow (\text{Mat}(m, n; \mathbb{C}), 0)$$

such that $(X, 0)$ appears as the preimage of the generic determinantal variety

$$M_{m,n}^s := \{B \in \text{Mat}(m, n; \mathbb{C}) : \text{rank } B < s\} \subset \text{Mat}(m, n; \mathbb{C})$$

under this map

$$(X, 0) = A^{-1}(M_{m,n}^s, 0),$$

and $(X, 0)$ has expected codimension $\text{codim}(X, 0) = \text{codim } M_{m,n}^s$.

By definition a deformation of a determinantal singularity comes from a perturbation of its matrix A as a map germ. The condition on $(X, 0)$ to have expected codimension assures that the induced family for the singularity is flat. Thus also the notions of deformation for (isolated) CMC2 and matrix singularities agree.

If a determinantal singularity $(X, 0)$ is isolated, it also automatically is an *EIDS* in the sense of Ebeling and Gusein-Zade [3]. Recall that the varieties

$$\{0\} = M_{m,n}^0 \subset M_{m,n}^1 \subset \dots \subset M_{m,n}^{\min\{m,n\}} \subset \text{Mat}(m, n; \mathbb{C})$$

give a canonical Whitney stratification of the space of $(m \times n)$ -matrices.

Definition 1.3. *A determinantal singularity $(X, 0) \subset (\mathbb{C}^N, 0)$ given by a matrix $A : (\mathbb{C}^N, 0) \rightarrow M_{m,n}$ as $A^{-1}(M_{m,n}^s)$ is an essentially isolated determinantal singularity (EIDS), if the map A is transverse to all strata of $M_{m,n}^s$ in a punctured neighborhood of the origin.*

In general, determinantal singularities do not admit smoothings, but only stabilizations. These are deformations coming from a perturbations A_ε of the defining matrix such that considered as a map, A_ε is transversal to all strata $M_{m,n}^s$ of $\text{Mat}(m, n; \mathbb{C})$. From this it is easy to see that a determinantal singularity $(X, 0) \subset (\mathbb{C}^N, 0)$ of type (m, n, s) admits a smoothing, if and only if

$$N < \text{codim } M_{m,n}^{s-1},$$

so that we have enough degrees of freedom to move the image of A away from the lower dimensional strata.

Definition 1.4. *Let $0 \in B \subset \mathbb{C}^N$ be a Milnor ball for a representative X of a smoothable isolated determinantal singularity $(X, 0) \subset (\mathbb{C}^N, 0)$ given by a matrix $A : B \rightarrow M_{m,n}$ as $X = A^{-1}(M_{m,n}^s)$. Let*

$$A_\varepsilon : B \rightarrow M_{m,n}$$

be a stabilization of A . The space $X_\varepsilon = A_\varepsilon^{-1}(M_{m,n}^s)$ is the Milnor fiber of $(X, 0)$. The generators of the homology groups $H_i(X_\varepsilon)$ are the vanishing cycles and the reduced Euler characteristic $\overline{\chi}(X_\varepsilon)$ the vanishing Euler characteristic of $(X, 0)$.

Throughout this paper we will only use homology and cohomology with integer coefficients. Hence we will omit them from the notation and just write $H_q(X)$ for $H_q(X; \mathbb{Z})$ and vice versa in cohomology.

One can use the theory of versal unfoldings for the map germ A to show that the diffeomorphism type of the Milnor fiber is unique. Thus the homology groups of X_ε and its invariants are in fact invariants of the singularity $(X, 0)$ itself.

Remark 1.5. *Let $(X, 0) \subset (\mathbb{C}^{n+d}, 0)$ be an isolated complete intersection singularity (ICIS) of codimension d . $(X, 0)$ can be seen as a determinantal singularity*

of type $(d, 1, 1)$. It is known [9] that its Milnor fiber X_ε is homotopic to a bouquet of spheres

$$X_\varepsilon \cong S^n \vee \dots \vee S^n$$

of real dimension n . Hence besides b_0 there is only the middle Betti number $b_n(X_\varepsilon)$, which is nonzero. It is also known as the Milnor number $\mu(X, 0)$ of the singularity.

Greuel and Steenbrink showed in [8] that the Betti numbers b_i of the Milnor fiber X_ε of any smoothable isolated singularity $(X, 0) \subset (\mathbb{C}^N, 0)$ must be zero in the range $0 < i \leq \dim X - \operatorname{codim} X$. For a smoothable ICMC2 singularity of dimension $n = \dim(X, 0) \geq 2$ this means that there are two possibly nonzero Betti numbers b_n and b_{n-1} of the Milnor fiber X_ε . However it is also shown in [8], that for surfaces b_1 is always zero. It is an open question, whether the first homology group is really zero or if there is torsion, but in general the vanishing cycles for ICMC2 surface singularities were not expected to behave very different from the complete intersection case.

For threefolds there are examples, for which this is no longer the case, as was first observed by James Damon and Brian Pike in [2]. Using Macaulay2, they computed the reduced Euler characteristic

$$-\overline{\chi}(X_\varepsilon) = b_3 - b_2$$

of the Milnor fiber and showed that it is negative for certain examples from the list of simple ICMC2 singularities in [4].

Methods for the computation of the Euler characteristic have also been developed in [3], [6] and [14]. Those using polar multiplicities can also be effectively implemented in Singular to determine the Euler characteristic of given examples. However neither of the methods is suited to compute the two Betti numbers or even the distinct homology groups independently.

This was first done by Anne Fröhbis Krüger and the author in [5] for the simple ICMC2 threefolds. The key tool in that paper was the *Tjurina modification*, which we will briefly review in the next section. Although the original intention in [5] was to explain the vanishing topology of threefolds, the ideas and techniques are also applicable in any other dimension greater than zero and beyond the ICMC2 case for determinantal singularities in general, see e.g. [13].

1.1 Tjurina transform for ICMC2 singularities

Let $(X_0, 0) \subset (\mathbb{C}^N, 0)$ be a determinantal singularity of type (m, n, t) given by the matrix $A \in \operatorname{Mat}(m, n; \mathbb{C}\{\underline{x}\})$. Consider A as a map germ

$$A : (\mathbb{C}^N, 0) \rightarrow (M_{m,n}, 0).$$

The generic determinantal variety $M_{m,n}^t$ is singular along the set $M_{m,n}^{t-1}$. Let

$$L : M_{m,n}^t \dashrightarrow \operatorname{Gr}(t-1, n), \quad B \mapsto \operatorname{span}(B^T)$$

be the rational map to the Grassmannian of $(t-1)$ -dimensional subspaces of \mathbb{C}^m , mapping a matrix B to the plane spanned by the image of its transpose. It is well defined on $M_{m,n}^t$ away from its singular locus $M_{m,n}^{t-1}$. The blowup of L

$$\hat{M}_{m,n}^t := \overline{\Gamma_L(M_{m,n}^t \setminus M_{m,n}^{t-1})} \subset \text{Mat}(m, n; \mathbb{C}) \times \text{Gr}(t-1, n)$$

is as usual defined to be the closure of the graph Γ_L of L over the regular part. It is a resolution

$$\rho : \hat{M}_{m,n}^t \rightarrow M_{m,n}^t$$

of $M_{m,n}^t$ with exceptional locus $\text{Gr}(t-1, n)$ over every point of $M_{m,n}^{t-1}$.

Definition 1.6. For a determinantal singularity $(X_0, 0) \subset (\mathbb{C}^N, 0)$ of type (m, n, t) given by a matrix A , the Tjurina transform

$$(Y_0, V) \subset (\mathbb{C}^N \times \text{Gr}(t-1, n), \{0\} \times \text{Gr}(t-1, n))$$

is defined as the fiber product

$$\begin{array}{ccccc} X_0 \times_{M_{m,n}^t} \hat{M}_{m,n}^t & \longrightarrow & \hat{M}_{m,n}^t & & \\ \pi \downarrow & & \rho \downarrow & \searrow \hat{L} & \\ X_0 & \xrightarrow{A} & M_{m,n}^t & \xrightarrow{L} & \text{Gr}(t-1, n) \end{array}$$

For ICMC2 singularities the target Grassmannian is always

$$\text{Gr}(t-1, t) \cong \mathbb{P}^{t-1},$$

where we identify a $(t-1)$ -plane with the class of its normal vector in the dual space. The equations for the Tjurina transform take a surprisingly simple form, see [5], corollary 3.3. Let $\underline{x} = (x_1, \dots, x_N)$ be the local coordinates of \mathbb{C}^N and $(s_1 : \dots : s_t)$ the homogeneous coordinates of $\mathbb{P}^{(t-1)}$. If $A = (a_{i,j})_{i,j}$ was the matrix describing $(X_0, 0)$, then Y_0 is the zero locus of the equations

$$\begin{pmatrix} f_0 \\ \vdots \\ f_t \end{pmatrix} = \begin{pmatrix} a_{1,1}(\underline{x}) & \cdots & a_{1,t}(\underline{x}) \\ \vdots & & \vdots \\ a_{t+1,1}(\underline{x}) & \cdots & a_{t+1,t}(\underline{x}) \end{pmatrix} \cdot \begin{pmatrix} s_0 \\ \vdots \\ s_t \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \quad (1)$$

in $\mathbb{C}^N \times \mathbb{P}^{(t-1)}$. The excetional set V of the projection $\pi : Y_0 \rightarrow X_0$ is by this construction always the whole $\mathbb{P}^{(t-1)}$. Hence Y_0 might not be equidimensional if the Cohen-Macaulay type of $(X_0, 0)$ was too big. However counting dimensions one easily verifies that Y_0 is a local complete intersection iff $\dim(X_0, 0) \geq t-1$.

Because any deformation of an ICMC2 singularity $(X_0, 0)$ comes from a perturbation of the defining matrix, Tjurina modification is also applicable in family. This basically means, we take the Tjurina transform Y of a representative X of the total space of a deformation of $(X_0, 0)$, see [5], construction 3.6.

Thus every deformation of $(X_0, 0)$ over a base B canonically induces a family

$$\begin{array}{ccc} Y_0 & \hookrightarrow & Y \\ \pi_0 \downarrow & & \downarrow \pi \\ X_0 & \hookrightarrow & X \\ \downarrow & & \downarrow \varepsilon \\ \{0\} & \hookrightarrow & B \end{array}$$

with the Tjurina transform Y_0 as a special fiber. If $\dim(X_0, 0) \geq t-1$, i.e. if Y_0 is a local complete intersection, this family is automatically flat ([5], proposition 3.9).

In this setup, we can study deformations of $(X_0, 0)$ via the deformations of (Y_0, V) sitting over it. Moreover in a smoothing of $(X_0, 0)$, the induced map

$$\pi_\varepsilon : Y_\varepsilon \rightarrow X_\varepsilon$$

on the fibers over $\varepsilon \neq 0$ is always an isomorphism ([5], proposition 3.8) – a direct consequence of the fact that the matrix $A(x)$ cannot degenerate on smooth points $x \in X_\varepsilon$.

1.2 Previously known results and main theorem

The main theorem about the topology from [5] can be summarized as follows:

Theorem 1.7. ([5], theorem 4.4) *Let $(X_0, 0) \subset (\mathbb{C}^5, 0)$ be an ICMC2 threefold singularity of Cohen-Macaulay type 2 such that the Tjurina transform (Y_0, V) has at most isolated singularities. Then the Betti numbers of the Milnor fiber X_ε are given by*

$$b_0 = 1, \quad b_1 = 0, \quad b_2 = 1, \quad b_3 = r,$$

where r is the sum of the Milnor numbers of the ICIS in Y_0 .

We will generalize this result in two directions. First of all the ideas and the proof for theorem 1.7 in [5] carry over almost literally to the surface case.

Theorem 1.8. *Let $(X_0, 0) \subset (\mathbb{C}^4, 0)$ be an ICMC2 surface singularity of Cohen-Macaulay type 2 such that the Tjurina transform (Y_0, V) has at most isolated singularities. Then the Milnor fiber X_ε is simply connected. The second homology group splits into*

$$H_2(X_\varepsilon) \cong \mathbb{Z}^r \oplus \mathbb{Z} \tag{2}$$

where all cycles of the first summand come from the ICIS in the Tjurina transform and r is the sum of their Milnor numbers.

We give a sketch of the proof in order to also recall the key arguments.

Proof. The exceptional set $V = \{0\} \times \mathbb{P}^1$ of the Tjurina transform Y_0 is a strong deformation retract of Y_0 . Let $B = \bigcup_{i=1}^N$ be a union of Milnor balls around the singular points p_i of Y_0 . For $q > 0$ we have isomorphisms

$$H_q(V) \cong H_q(Y_0) \cong H_q(Y_0, B)$$

and if $q = 2$, $H_2(Y_0, B)$ is freely generated by a relative cycle, which can be represented by the exceptional set $V \cong \mathbb{P}^1$ with the interiors of the Milnor balls B_i cut out from it. It is a sphere with holes and the boundary consists of circles located in the links of the ICIS of Y_0 .

When we now pass to a smoothing Y_δ of Y_0 , we do not change the topology outside the Milnor balls B_i . Glueing in the local Milnor fibers $F_i \subset B_i$, we obtain the following long exact sequence

$$0 \longrightarrow H_2(\bigcup_i F_i) \longrightarrow H_2(Y_\delta) \longrightarrow H_2(Y_\delta, \bigcup_i F_i) \longrightarrow H_1(\bigcup_i F_i) \longrightarrow \dots$$

By excision we have $H_2(Y_\delta, \bigcup_i F_i) \cong H_2(Y_0, B) \cong \mathbb{Z}$. The zero on the left shows that there are no relations among the vanishing cycles in the local Milnor fibers F_i . On the right, the term $H_1(\bigcup_i F_i)$ vanishes because of Hamms result [9]. We deduce the desired splitting. \square

For ICMC2 singularities $(X_0, 0)$ for which the Tjurina transform is smooth, the Milnor fiber is diffeomorphic to (Y_0, V) . Consequently if we let

$$L : X_\varepsilon \rightarrow \mathbb{P}^1, \quad x \mapsto \text{span } A_\varepsilon^T(x),$$

be the regular map on the Milnor fiber given by the deformed matrix A_ε , then a generator of $H_2(X_\varepsilon)$ is given by the fundamental class of a differentiable section $l : \mathbb{P}^1 \rightarrow X_\varepsilon$ of L , i.e. a map l such that $L \circ l = \text{Id}_{\mathbb{P}^1}$.

In general the existence of such a section is hard to prove. But from the proof of theorem 1.8 it is evident that the generator of the second summand of the splitting (2), or just the generator of the second homology group in theorem 1.7, is “coming from” the exceptional set. To make this more precise, we give the following definition.

Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be a determinantal singularity of type (m, n, t) given by a matrix A and

$$A_\varepsilon : B \rightarrow M_{m,n}$$

a stabilization of A defined on some Milnor ball $B \subset \mathbb{C}^N$ for $(X, 0)$. Because A_ε is transverse to all the strata of $M_{m,n}^s$, the Tjurina transform $Y_\varepsilon \subset B \times \text{Gr}(t-1, n)$ of $X_\varepsilon = A_\varepsilon^{-1}(M_{m,n}^s)$ is a smooth compact manifold with corners (Recall that Y_ε is isomorphic to X_ε in case $(X, 0)$ was smoothable). By abuse of notation, let

$$L : Y_\varepsilon \subset B \times \text{Gr}(t-1, n) \rightarrow \text{Gr}(t-1, n)$$

be the projection to the Grassmannian. Consider the image $G \subset H^\bullet(Y_\varepsilon)$ of the induced map

$$L^* : H^\bullet(\text{Gr}(t-1, n)) \rightarrow H^\bullet(Y_\varepsilon)$$

in cohomology.

Definition 1.9. A cycle $[\sigma] \in H_\bullet(Y_\varepsilon)$ is said to be horizontal, if the cap product $g \cap [\sigma]$ is zero for all $g \in G = L^*(H^\bullet(\text{Gr}(t-1, n)))$. We also write

$$[\sigma] \in G^\perp.$$

All other cycles in $H_\bullet(Y_\varepsilon)$ are called vertical. We also say they are sitting over the Grassmannian.

Corollary 1.10. Let X_ε be the Milnor fiber of an ICMC2 singularity $(X_0, 0) \subset (\mathbb{C}^{n+2}, 0)$ of dimension $n = 2$ or 3 and of Cohen-Macaulay type $t = 2$ with only isolated singularities in the Tjurina transform. Then the homology of X_ε splits into

$$H_\bullet(X_\varepsilon) \cong G^\perp \oplus \mathbb{Z},$$

where the second summand lives in degree 2 and the cap product with $L^*([\mathbb{P}^1]^\vee)$, the pullback of the dual of the fundamental class of \mathbb{P}^1 , is a perfect pairing.

The main goal of this paper is to extend theorem 1.7, theorem 1.8 and corollary 1.10 to the case of arbitrary ICMC2 singularities of Cohen-Macaulay type $t = 2$ and dimension 2 or 3, i.e. we also allow nonisolated singularities in the Tjurina transform.

Theorem 1.11. (Main theorem) Let X_ε be the Milnor fiber of an ICMC2 singularity $(X_0, 0) \subset (\mathbb{C}^{n+2}, 0)$ of dimension $n = 2$ or 3 and Cohen-Macaulay type $t = 2$ given by a matrix $A \in \text{Mat}(3, 2; \mathbb{C}\{\underline{x}\})$ and its perturbation A_ε . Let

$$L : X_\varepsilon \rightarrow \mathbb{P}^1, \quad x \mapsto \text{span } A_\varepsilon^T(x).$$

The Milnor fiber X_ε is simply connected and the homology of X_ε splits into

$$H_\bullet(X_\varepsilon) \cong G^\perp \oplus \mathbb{Z}.$$

The cap product with $L^*(H^2(\mathbb{P}^1))$ gives a perfect pairing of the vertical cycles with $H^2(\mathbb{P}^1) \cong \mathbb{Z}$. If $n = 3$, then $H_2(X_\varepsilon) \cong \mathbb{Z}$ consists of the vertical cycles only.

Since for any given example the Euler characteristic $\chi(X_\varepsilon)$ can be computed by e.g. the polar multiplicities [14] of $(X_0, 0)$, we obtain the following corollary.

Corollary 1.12. Let X_ε be the Milnor fiber of an ICMC2 threefold singularity of Cohen-Macaulay type $t = 2$. The Betti numbers of X_ε can be computed as

$$\begin{aligned} b_0 &= 1 \\ b_2 &= 1 \\ b_3 &= -\chi(X_\varepsilon) + 2 \\ b_k &= 0 \quad \text{for } k \notin \{0, 2, 3\}, \end{aligned}$$

1.3 An example and outline of the proof

To illustrate the ideas of the proof of theorem 1.11, we give an example of an ICMC2 threefold singularity with non-isolated singular locus in the Tjurina transform.

Let $(X_0, 0) \subset (\mathbb{C}^5, 0)$ be given by the matrix

$$\begin{pmatrix} v & x \\ w & y \\ -2xy & v^2 + w^2 + z^2 \end{pmatrix} \quad (3)$$

and consider the smoothing obtained by perturbing the lower left entry with a constant δ . We denote the homogeneous coordinates of \mathbb{P}^1 by $(s_1 : s_2)$. Then the equations for the Tjurina transform $(Y_0, V) \subset (\mathbb{C}^5 \times \mathbb{P}^1, \{0\} \times \mathbb{P}^1)$ and its deformation by δ are

$$\begin{pmatrix} v & x \\ w & y \\ -2xy - \delta & v^2 + w^2 + z^2 \end{pmatrix} \cdot \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = 0. \quad (4)$$

Let us look at the first chart $\{s_1 \neq 0\}$. We write $s = s_2/s_1$ for the corresponding standard affine coordinate. The equations from the first two rows read

$$v = -s \cdot x, \quad w = -s \cdot y.$$

Substituting this in the equation from the last row, we obtain a hypersurface

$$h = s^3 \cdot x^2 + s^3 \cdot y^2 - 2xy + s \cdot z^2,$$

which is perturbed by a constant δ . We can interpret this as a quadratic form Q_s in (x, y, z) parametrized by s and write it in the standard matrix form:

$$h = Q_s(x, y, z) = \begin{pmatrix} x & y & z \end{pmatrix} \cdot \begin{pmatrix} s^3 & -1 & 0 \\ -1 & s^3 & 0 \\ 0 & 0 & s \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \delta$$

Any quadratic form should of course be diagonalized. To do this, we introduce new coordinates

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} := \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

in which our hypersurface equation takes the form

$$h = Q_s(\tilde{x}, \tilde{y}, \tilde{z}) = \begin{pmatrix} \tilde{x} & \tilde{y} & \tilde{z} \end{pmatrix} \cdot \begin{pmatrix} s^3 + 1 & 0 & 0 \\ 0 & s^3 - 1 & 0 \\ 0 & 0 & s \end{pmatrix} \cdot \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \delta \quad (5)$$

This is a family of A_1 -surface singularities, which degenerates as s approaches one of the seven values

$$s \in \sqrt[6]{1} \cup \{0\}.$$

Now it is clear that in this chart the Tjurina transform Y_0 is singular along the whole exceptional set V , the s -axis in this chart.

Let $L : \mathbb{C}^5 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be the standard projection, i.e. in this chart the projection to the s -axis. If we restrict h to a general transversal slice to V given by the hypersurface $\{L = c\}$ for a general $c \in \mathbb{C}$, we obtain the *transversal singularity*, denoted by Y_0^\natural

$$h|_{\{L=c\}} = (c^3 + 1)\tilde{x}^2 + (c^3 - 1)\tilde{y}^2 + c\tilde{z}^2 = \delta$$

and a smoothing induced by the perturbation with the constant δ . This transversal singularity is isolated and of type A_1 .

For $\delta \neq 0$ we see a vanishing cycle $[\sigma]$ in the Milnor fiber

$$Y_\delta^\natural = \{h = \delta\} \cap \{L = c\}.$$

of the transversal singularity. It lives in the second homology group $H_2(Y_\delta^\natural)$ and can be represented by a 2-sphere. This is a candidate for further contributions of the second homology group of

$$Y_\delta \subset \mathbb{C}^5 \times \mathbb{P}^1,$$

the fiber over δ in the given deformation, and hence for the Milnor fiber X_ε of $(X_0, 0)$. Whether or not $[\sigma]$ is nonzero as an element of $H_2(Y_\delta)$ depends on the inclusion

$$Y_\delta^\natural \subset Y_\delta.$$

To shed some light on this question, let us observe the behaviour close to the degeneracy points

$$K := \left\{ (\tilde{x}, \tilde{y}, \tilde{z}, s) : \tilde{x} = \tilde{y} = \tilde{z} = 0, s \in \sqrt[6]{1} \cup \{0\} \right\}.$$

The analytic type of the singularity h at either of these points is what D. Siersma calls the D_∞ -singularity, a.k.a. the *Whitney umbrella*. For any $p \in K$ we can choose a Milnor ball $B = B(p)$ for the singularity of h around p and a value $c \in \mathbb{C}$ for the transversal singularity sufficiently close to $s(p)$ such that the intersection B with the hyperplane $\{L = c\}$ is nonempty. D. Siersma shows in [16], proposition 3.8:

For the D_∞ singularity of dimension n the pair of Milnor fibers $(Y_\delta \cap B, Y_\delta^\natural \cap B)$ is homotopy equivalent to the pair of spheres

$$(S^n, S^{n-1}),$$

where $S^{n-1} \hookrightarrow S^n$ is the standard equatorial embedding.

Let $W \subset \mathbb{C}$ be the complement of some small discs around the special points in $K \subset \mathbb{C}$. Then for $\delta > 0$ small enough

$$L : Y_\delta \cap L^{-1}(W) \rightarrow W \quad (6)$$

is a fiber bundle with fiber Y_δ^\cap . This means we can freely move the equator of all the vanishing cycles coming from the seven D_∞ points and connect all half spheres globally. The affine part of Y_δ is therefore homotopic to a bouquet of spheres:

$$Y_\delta \setminus \{s_1 \neq 0\} \cong \underbrace{S^3 \vee \cdots \vee S^3}_{2 \cdot 7 - 1 \text{ times}}, \quad (7)$$

with each of their equators being homologous to the vanishing cycle S^2 of any of the transversal Milnor fibers.

To complete the picture, let us look at the other chart $\{s_2 \neq 0\}$. We denote the corresponding affine coordinate of \mathbb{P}^1 by $t = s_1/s_2$. Again the equations for the first two rows of the matrix allow us to substitute in the equation of the third row and we obtain the perturbation of a hypersurface equation:

$$h := v^2 + w^2 - 2t^3 \cdot vw + z^2 = \delta \cdot t.$$

Regarding this as a quadratic form Q_t in (v, w, z) parametrized by t and diagonalizing as before, we obtain

$$\begin{pmatrix} \tilde{v} & \tilde{w} & \tilde{z} \end{pmatrix} \cdot \begin{pmatrix} 1+t^3 & 0 & 0 \\ 0 & 1-t^3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \tilde{v} \\ \tilde{w} \\ \tilde{z} \end{pmatrix} = \delta \cdot t.$$

We do recover the six degeneracy values for t of the quadratic form at the six roots of unity. However, Q_t does not degenerate at the point $(0, \infty) \in \mathbb{C}^5 \times \mathbb{P}^1$, the origin in this chart. Hence we can make an analytic change of coordinates around this point such that the local equation h for Y_0 at $(0, \infty)$ is just an A_∞ singularity:

$$h = x^2 + y^2 + z^2 : (\mathbb{C}^4, 0) \rightarrow (\mathbb{C}, 0).$$

But note that we do not perturb by a constant, but by $\delta \cdot t$. This means, the transversal slice over $\{t = 0\}$ does not deform! We have

$$Y_\infty^\cap := Y_0 \cap \{t = 0\} = Y_\delta \cap \{t = 0\}.$$

The set Y_∞^\cap is what we call an *axis of the deformation* and its intersection with the exceptional set V the *axis point* $(0, \infty)$.

Being a representative of the germ of an isolated singularity, Y_∞^\cap is a contractible fiber in the family given by $L : Y_\delta \rightarrow \mathbb{P}^1$. For the vanishing cycle $[\sigma]$ of the transversal Milnor fiber, the equator of all the 3-spheres generating the homology of $Y_\delta \setminus \{s_1 \neq 0\}$, this gives one more opportunity to close. Hence we obtain:

$$H_3(Y_\delta) \cong \mathbb{Z}^{14}$$

is freely generated by 3-spheres.

But Y_δ is not homotopic to a bouquet of spheres, as one might think at this point. There is a nontrivial cycle in $H_2(Y_\delta)$ sitting over \mathbb{P}^1 , which is constructed as follows.

Recall (6) that $Y_\delta \cap L^{-1}(W)$ had the structure of a fiber bundle over W by means of L . The fiber Y_δ^\cap is homotopic to S^2 , while the base W has the homotopy type of a bouquet of 6 circles. Obstruction theory tells us, that up to homotopy there is a unique continuous section

$$l : W \rightarrow Y_\delta \cap L^{-1}(W)$$

of L . Because over $\infty \in \mathbb{P}^1$ we only glue in a contractible fiber, we can certainly extend l to $W \cup \{\infty\}$. Let $B = \overline{\mathbb{P}^1 \setminus (W \cup \{\infty\})}$ be the closure of the complement of $W \cup \{\infty\}$. The image of l defines a unique relative cycle $[l]$ in $H_2(Y_\delta, L^{-1}(B))$, whose boundary consists of seven circles in the links of the local Milnor fibers of the D_∞ points. At every such point p_i we can choose local coordinates, in which Y_δ is in the normal form

$$s \cdot x^2 + y^2 + z^2 = \delta.$$

We can extend l by glueing

$$s \mapsto (s, x, y, z) = (s, 0, 0, \sqrt{\delta})$$

to the respective part of the boundary $\partial[l]$ of $[l]$ and obtain a global section $l' : \mathbb{P}^1 \rightarrow Y_\delta$. Consider the long exact sequence of the pair $(Y_\delta, L^{-1}(B))$:

$$H_2(L^{-1}(B)) \xrightarrow{\iota} H_2(Y_\delta) \longrightarrow H_2(Y_\delta, L^{-1}(B)) \xrightarrow{\partial^*} H_1(L^{-1}(B))$$

The existence of the local extensions of l tells us that $\partial^*([l]) = 0$ and also $H_2(Y_\delta \cap L^{-1}(B)) = 0$. Hence $H_2(Y_\delta)$ is freely generated by $[l']$, the image of the fundamental class of \mathbb{P}^1 under l . It is evident that

$$L^* : H^2(\mathbb{P}^1) \rightarrow H^2(Y_\delta) = \text{Hom}(H_2(Y_\delta), \mathbb{Z})$$

is an isomorphism.

Because the deformation we started with was a smoothing of $(X_0, 0)$, the spaces Y_δ and X_δ are naturally isomorphic and we're done with the determination of the homology groups of the Milnor fiber of $(X_0, 0)$.

We will now outline the proof of the main theorem 1.11 using this example. It is widely inspired by the work of D. Siersma and M. Tibar on the vanishing topology of projective hypersurfaces [18], in the way we piece together the global picture from local computations and the role played by the axis point.

Step I: Study line singularities, which are local complete intersections. In general the singular locus $V = \{0\} \times \mathbb{P}^1$ of the Tjurina transform $Y_0 \subset \mathbb{C}^5 \times \mathbb{P}^1$ will

consist of a Zariski open set U , over which the projection L to \mathbb{P}^1 induces the structure of a fiber bundle with fiber Y_0^\natural , the transversal singularity. Its Milnor fiber Y_δ^\natural is well defined up to diffeomorphism. This is done in section 2.3.

Then we will treat the special points, i.e. the complement of U . In the above example we saw that the vanishing cycle $[\sigma]$ of the transversal singularity became homologous to zero in the local Milnor fibers of the D_∞ singularities. But in the general case of arbitrary line singularities which are complete intersections, there is no reason for this to hold. Consider for example the F_1A_3 singularity from De Jongs list [11]:

$$f = xz^2 + y^2z = z \cdot (xz + y^2).$$

He shows that its Milnor fiber F is homotopy equivalent to S^1 . If we find such a singularity in the Tjurina transform of an ICMC2 surface singularity or a double suspension of it in the Tjurina transform of a threefold, then there are cycles of the transversal Milnor fiber F^\natural , which are not homologous to zero in F .

It turns out that the important property we need, is the fact that any vanishing cycle of degree $(n-1)$ of the Milnor fiber F of a complete intersection line singularity can be represented by a cycle in the transversal Milnor fiber F^\natural . This is done in section 2.4, where we give a description of how the local Milnor fiber of those singularities is connected to its transversal Milnor fiber (corollary 2.9 and theorem 2.11 for the threefolds and respectively 2.10 and 2.13 for the surface case).

Step II: The role of the axis point. In section 3.1 we show that for deformations of an ICMC2 singularity $(X_0, 0)$ of dimension n and its Tjurina transform (Y_0, V) coming from a perturbation of the defining matrix A with a general constant matrix B of rank 1, a *generic rank 1 perturbation*, we always have an axis Y_∞^\natural and an axis point $(0, \infty) \in V$. For the fiber Y_δ of the Tjurina transform in such a deformation, the connectivity of local Milnor fibers F of complete intersection line singularities with their transversal Milnor fibers F^\natural will imply that all homology of degree $n-1$ of $Y_\delta \setminus Y_\infty^\natural$ is concentrated in the transversal Milnor fiber Y_δ^\natural . When glueing in the fiber Y_∞^\natural of L over ∞ , all the cycles in Y_δ^\natural collapse.

Step III: Putting together the global picture. In the last part, section 3.3, we use Mayer-Vietoris arguments to compute the homology groups of Y_δ for a general rank 1 perturbation (theorem 3.4): While the vanishing cycles of Y_δ^\natural and hence also all $(n-1)$ -cycles of the local Milnor fibers of the special points are homologous to zero in Y_δ , adding Y_∞^\natural to $Y_\delta \setminus Y_\infty^\natural$ will also give rise to a new 2-cycle sitting over \mathbb{P}^1 in the sense of definition 1.4. This finally leads to the proof of the main theorem 1.11, in which we pass from a general rank 1 perturbation, for which neither Y_δ nor X_δ are necessarily smooth, to a smoothing of $(X_0, 0)$.

2 Topology of line singularities

Definition 2.1. A singularity $(Y_0, 0) \subset (\mathbb{C}^N, 0)$ is called a *line singularity*, if the singular locus $V = \text{Sing}(Y_0)$ is the germ of a line in \mathbb{C}^N at 0.

Curves cannot have line singularities – unless they are a multiple line themselves. In this section, we will therefore always assume $n = \dim(Y_0, 0) \geq 2$.

Let $(Y_0, 0) \subset (\mathbb{C}^N, 0)$ be a line singularity, which is a complete intersection of codimension d given by the equations $f_1 = \cdots = f_d = 0$. For line singularities there is in general no unique smoothing. But if we consider the defining equations f_i as components of a map germ

$$f : (\mathbb{C}^N, 0) \rightarrow (\mathbb{C}^d, 0),$$

then for a chosen Milnor ball B for $(Y_0, 0)$, the preimage of a regular value $c \in \mathbb{C}^d$ of f on B sufficiently close to 0 is unique up to diffeomorphism. This is what we will refer to as the (local) Milnor fiber of the line singularity $(Y_0, 0)$.

There is a well-known trick to reduce the problem to a Milnor fiber of one holomorphic function on a controlled ambient space, see e.g. [9]. Let $0 \in U \subset \mathbb{C}^N$ be a neighborhood of the origin, on which all the f_i are defined. Consider the map

$$\mathbb{P}f : U \setminus Y_0 \rightarrow \mathbb{P}^{d-1}, \quad x \mapsto (f_1(x) : \cdots : f_d(x))$$

and choose a regular value $p \in \mathbb{P}^{d-1}$ for $\mathbb{P}f$. After a change of coordinates of \mathbb{P}^{d-1} , which corresponds to a new \mathbb{C} -linear combination of the generators f_i , we can assume that $p = (1 : 0 : \cdots : 0)$. Then the closure of its preimage in $U \subset \mathbb{C}^N$ is given by

$$Y^* = \{x \in U : f_2(x) = \cdots = f_d(x) = 0\}. \quad (8)$$

Lemma 2.2. *The singular locus of Y^* is contained in the singular locus of Y .*

Proof. (cf. [9], Lemma 1.1 or Lemma 2.2) Outside Y_0 the space Y^* is already smooth. If Y^* had a singular point $p \in Y_0$, this means that the jacobian of (f_2, \dots, f_d) does not have full rank at p . But then also the jacobian of (f_1, f_2, \dots, f_d) cannot have full rank and hence p is a singular point of Y_0 as well. \square

We rename the first function f_1 by f . Without loss of generality we can assume, that the singular line $(V, 0) = (\text{Sing}(Y), 0)$ is just the germ of the first coordinate axis of \mathbb{C}^N . This will be the *standard situation*, from which we will proceed for this section:

$$f : (Y^*, 0) \subset (\mathbb{C}^N, 0) \rightarrow (\mathbb{C}, 0), \quad (9)$$

$$(\text{Sing}(Y^*), 0) = (\{x_2 = \cdots = x_N = 0\}, 0). \quad (10)$$

Since we are primarily interested topological questions about the singularity, we will use Whitney stratifications to provide the setup for applications of

the first Thom isotopy lemma. We may assume that Y^* admits a Whitney stratification by the strata

$$(Y^* \setminus Y_0, Y_0 \setminus V, V \setminus \{0\}, \{0\}) \quad (11)$$

sufficiently close to the origin. The last stratum $\{0\}$ might however be optional.

2.1 The polar curve

Besides the Whitney stratification there is one more thing we need to take into account. Let

$$L : \mathbb{C}^N \rightarrow \mathbb{C}, (x_1, \dots, x_N) \mapsto x_1$$

be the projection to the first coordinate axis. The polar locus of f with respect to L on Y^* is defined as

$$\Gamma(f, L) = \overline{\{x \in Y^* \setminus Y_0 : dL(x), df(x) \text{ are linearly dependent in } \Omega_{Y^*}^1\}}, \quad (12)$$

where $\overline{}$ denotes the closure. The polar locus can be very nasty. However for most choices of the projection L we get a reasonable control over Γ .

Lemma 2.3. *For $a = (a_2, \dots, a_N)$ let*

$$L_a : \mathbb{C}^N \rightarrow \mathbb{C}, (x_1, \dots, x_N) \mapsto x_1 - \sum_{i=2}^N a_i \cdot x_i$$

be the projection bent by a . There exists a dense set $\Omega \subset \mathbb{C}^{N-1}$ of values for a , such that for $a \in \Omega$ the polar locus $\Gamma(f, L_a) \subset Y^$ is either empty or an analytic curve, which is smooth outside Y_0 .*

Proof. This is a Bertini-type theorem. Consider the following incidence space

$$N^* := \{(x, a) \in Y^* \times \mathbb{C}^{N-1} : dL_a(x), df(x) \text{ are linearly dependent in } \Omega_{Y^*}^1\}.$$

It comes along with the two natural projections

$$\begin{array}{ccc} & N^* & \\ pr_1 \swarrow & & \searrow pr_2 \\ Y^* & & \mathbb{C}^{N-1} \end{array}$$

Over $Y^* \setminus Y_0$ the function f does have a full rank differential on TY^* and therefore $N^* \setminus pr_1^{-1}(Y_0)$ is a smooth manifold of complex dimension

$$\dim N^* = \dim Y^* + (N - 1) - (\dim Y^* - 1) = N.$$

Let a be a regular value of the projection pr_2 restricted to $N^* \setminus pr_1^{-1}(Y_0)$. Its preimage

$$pr_2^{-1}(\{a\}) \subset Y^* \times \{a\}$$

is either empty or an analytic curve, which is smooth outside $Y_0 \times \{a\}$. \square

We will in the following assume that L_a has been chosen according to Lemma 2.3. Then we readjust the coordinate system of \mathbb{C}^N in a way that $L_a = L = x_1$ is just the first coordinate function, i.e. the projection to the first axis.

Corollary 2.4. *Passing to a smaller representative of Y_0 if necessary, we can furthermore assume that the polar curve meets Y_0 only at points in V .*

2.2 The choice of a Milnor ball

Let $\rho : \mathbb{C}^N \rightarrow \mathbb{R}$ be the squared distance function from the origin and set $B_\varepsilon := \{\rho \leq \varepsilon\}$. For sufficiently small $\varepsilon > 0$ we may assume that

- ρ is a Whitney stratified submersion on $Y^* \cap B_\varepsilon$ with respect to the standard stratification (11).
- (cf. [9], Korollar 3.2) the function

$$\arg f : Y^* \setminus Y_0 \rightarrow S^1$$

has a differential which is linearly independent of $d\rho$ in T^*Y^* along $B_\varepsilon \cap (Y^* \setminus Y_0)$.

- the function f has no critical points on $B_\varepsilon \cap Y^*$ away from Y_0 .
- the polar curve Γ is either empty or it intersects $B_\varepsilon \cap V$ at most at the origin.

2.3 The Milnor fiber in the product case

In this section we will treat the case that

$$(Y^* \setminus Y_0, Y_0 \setminus V, V)$$

is already a Whitney stratification of Y^* at 0 and the polar curve Γ is empty.

Thom's first isotopy lemma yields that Y_0 is a product over V :

$$(Y_0, 0) \cong (Y_0^\natural \times V, 0), \tag{13}$$

where $(Y_0^\natural, 0)$ is the germ of the *transversal singularity*. This is an isolated singularity obtained from Y_0 by intersecting it with a hyperplane in general position, i.e. transversal to all strata at 0. Lemma 2.3 and corollary 2.4 show us, how to choose the equation for such a hyperplane.

We will show that the product structure (13) also holds for the Milnor fiber. To do so, it is more convenient to have a polydisc rather than a Milnor ball. Assume that the projection $L = x_1$ to the first axis is general and let

$$q = \sum_{i=2}^N \bar{x}_i \cdot x_i : \mathbb{C}^N \rightarrow \mathbb{R}$$

be the squared distance from V . According to [7], lemma 2.3, the map

$$(L, q) : Y_0 \setminus V \rightarrow V \times \mathbb{R}$$

is a submersion on a neighborhood U of the origin. We may choose $\alpha, \beta \in \mathbb{R}_{>0}$ small enough such that the polydisc

$$\Delta_{\alpha\beta} := \{q \leq \alpha^2\} \cap \{|L| \leq \beta\}$$

is contained in U .

Theorem 2.5. *In the above setup for fixed α and β there exists a $\delta > 0$ such that the map*

$$(f, L) : Y^* \cap \Delta_{\alpha\beta} \cap f^{-1}(D_\delta) \rightarrow D_\delta \times D_\beta \quad (14)$$

is a fiber bundle away from $Y_0 = Y^ \cap \{f = 0\}$.*

Definition 2.6. *The fiber of (14) over a general point is called the transversal Milnor fiber and denoted by F^\natural .*

Clearly for fixed $\delta > 0$ we have $F \cong F^\natural \times D_\beta$.

Proof. (of theorem 2.5) Since (L, q) was a submersion on $Y_0 \cap \Delta_{\alpha\beta}$, the horizontal part of the boundary

$$\partial_h(Y_0 \cap \Delta_{\alpha\beta}) := Y_0 \cap \{q = \alpha^2\} \cap \{|L| \leq \beta\}$$

is a fiber bundle over the closed disc D_β . Because it is compact, this property is preserved under small perturbations of f . Hence we can assume that

$$(f, L) : Y^* \cap f^{-1}(D_\delta) \cap \{q = \alpha^2\} \cap L^{-1}(D_\beta) \rightarrow D_\delta \times D_\beta$$

is a fiber bundle.

The absence of the polar curve assures that away from Y_0 we also find no critical points of (f, L) in the interior of $Y^* \cap \Delta_{\alpha\beta}$. Therefore (14) is a proper submersion away from Y_0 and hence a fiber bundle by Ehresmann's fibration theorem. \square

2.4 The Milnor fiber at a special point

We now treat the general case, i.e. we have a Whitney stratification of $Y^* \subset \mathbb{C}^N$ by the strata

$$(Y^* \setminus Y_0, Y_0 \setminus V, V \setminus \{0\}, \{0\})$$

and a possibly nonempty polar curve $\Gamma \subset Y^*$ which meets Y_0 at $\{0\}$. By passing to smaller representatives if necessary, we can always reduce to this setup.

Let B be a Milnor ball for Y_0 at 0. When we investigate the topology at the special points in the setting of the Tjurina-modification of an ICMC2 singularity, it is the part of the boundary $\Sigma = Y_0 \cap \partial B$ which is close to V , along which Y_0 connects to the remaining space. Therefore we will study mainly two objects in this section: The topology of the second boundary

$$\partial_2 F \subset \partial F,$$

which is the part of the boundary of the Milnor fiber F close to V , and the relative homology groups

$$H_q(F, \partial_2 F),$$

which determine how F is connected to $\partial_2 F$. The precise definition of the second boundary $\partial_2 F$ is given below.

2.4.1 The second boundary

In this section we will denote the boundaries of the spaces in question by

$$\Sigma^* := Y^* \cap \partial B, \quad \Sigma := Y_0 \cap \partial B, \quad S := V \cap \partial B$$

Along the points of S we find the product situation of the preceeding section for Y_0 . Thus theorem 2.5 is applicable along the whole circle. However we do need the slight modification to change L to

$$\tilde{L} : B \rightarrow \mathbb{C}, \quad x \mapsto \sqrt{\rho(x)} \cdot \exp(\sqrt{-1} \cdot \arg L),$$

with $\rho = |L|^2 + q$ the squared distance from the origin. The function \tilde{L} is not holomorphic, but approximates L as a differentiable function close to S . Repeating the arguments in the setup and proof of theorem 2.5 along the compact manifold S we obtain:

Corollary 2.7. *There exist $\alpha > 0$ and $\delta > 0$ sufficiently small with respect to α such that*

$$(f, \arg L) : \Sigma^* \cap \{q \leq \alpha^2\} \cap f^{-1}(D_\delta) \rightarrow D_\delta \times S^1 \quad (15)$$

is a smooth fiber bundle away from $\{f = 0\}$.

It is easy to see that the fiber of this fiber bundle is canonically diffeomorphic to the transversal Milnor fiber F^\natural .

Definition 2.8. *For α and δ as in corollary 2.7 the space*

$$\partial_2 F := \Sigma^* \cap \{q \leq \alpha^2\} \cap \{f = \delta\}$$

is called the second boundary of the Milnor fiber F and the monodromy from the fibration

$$\arg L : \partial_2 F \rightarrow S^1 \quad (16)$$

the vertical monodromy.

The topology of $\partial_2 F$ is completely determined by the topology of F^\natural and the monodromy operator of the Wang sequence of (16). The transversal Milnor fiber F^\natural comes from an ICIS of dimension $n - 1$, so it is $(n - 2)$ -connected. For $n \geq 3$ the Wang sequence splits into two parts

$$0 \longrightarrow H_n(\partial_2 F) \longrightarrow H_{n-1}(F^\natural) \xrightarrow{T_{n-1}-1} H_{n-1}(F^\natural) \longrightarrow H_{n-1}(\partial_2 F) \longrightarrow 0 \quad (17)$$

and

$$0 \longrightarrow H_1(\partial_2 F) \longrightarrow H_0(F^\heartsuit) \xrightarrow{\mathbf{T}_0 - \mathbf{1}} H_0(F^\heartsuit) \longrightarrow H_0(\partial_2 F) \longrightarrow 0 \quad (18)$$

where \mathbf{T}_\bullet is the monodromy operator of (16). Clearly, $\mathbf{T}_0 - \mathbf{1}$ in (18) is the zero map. Thus we proved the following.

Corollary 2.9. *Let $n = \dim(Y_0, 0) \geq 3$. The homology groups of $\partial_2 F$ have the following properties:*

1. $H_n(\partial_2 F)$ is a free subgroup of $H_{n-1}(F^\heartsuit)$.
2. Every cycle in $H_{n-1}(\partial_2 F)$ can be represented by a cycle in $H_{n-1}(F^\heartsuit)$.
3. $H_1(\partial_2 F)$ is free abelian of rank 1 and generated by a section of $\arg L$.
4. $\partial_2 F$ is connected.
5. All other homology groups are zero.

If $n = \dim(Y_0, 0) = 2$, the terms $H_{n-1}(\partial_2 F)$ from (17) and $H_1(\partial_2 F)$ in (18) come together. But the kernel of $\mathbf{T}_0 - \mathbf{1}$ is still free of rank 1 and hence there is a (non-canonical) splitting

$$H_1(\partial_2 F) \cong H'_1 \oplus \mathbb{Z} = \text{coker}(T_1 - \mathbf{1}) \oplus \ker(T_0 - \mathbf{1}). \quad (19)$$

We call $H'_1 = \text{coker } T_1 - \mathbf{1}$ the *transversal* or *horizontal* and the other summand $\mathbb{Z} = \ker T_0 - \mathbf{1}$ the *vertical* cycles of the second boundary $\partial_2 F$.

Corollary 2.10. *The homology groups of the second boundary $\partial_2 F$ of the Milnor fiber F of a complete intersection line singularity $(Y_0, 0)$ of dimension 2 have the following properties:*

1. $H_2(\partial_2 F)$ is a free subgroup of $H_1(F^\heartsuit)$.
2. Every cycle in $H_1(\partial_2 F)$ can be represented by a transversal cycle in $H_1(F^\heartsuit)$.
3. $H_1(\partial_2 F)$ splits into transversal and vertical cycles (19) and a generator of the latter is given by the fundamental class of a section of $\arg L$.
4. $\partial_2 F$ is connected.
5. All other homology groups are zero.

2.4.2 Connectivity of the Milnor fiber with the second boundary

Having described the topology of the second boundary we now turn to the question, how it connects with the Milnor fiber. We will first treat the case $n = \dim(Y_0, 0) \geq 3$ and modify the arguments for the surface case in the next section.

Theorem 2.11. *Let $n = \dim Y_0 \geq 3$. Then we have*

$$H_q(F, \partial_2 F) \cong \begin{cases} 0 & 2 < q < n \\ H_{q-1}(\partial_2 F) & q = 2 \\ 0 & 0 \leq q \leq 1 \end{cases} \quad (20)$$

where the isomorphisms are induced from the long exact sequence of the pair of spaces $(F, \partial_2 F)$.

The proof of theorem 2.11 follows closely the ideas of Dirk Siersma in his paper [17]. He proved it in the case of hypersurfaces, with possibly even more complicated singular locus, as the corollary of lemma 3.8, his “second variation sequence”¹. It picks up the idea of the original fibration by Milnor

$$\arg f : \Sigma^* \setminus \Sigma \rightarrow S^1, \quad (21)$$

where as before $\Sigma^* = Y^* \cap \partial B$ and Σ is the boundary of Y_0 . Hamm shows in [9], Satz 1.6, that this is a C^∞ -fiber bundle with open fibers. Moreover he proves that for $\delta > 0$ sufficiently small, (21) is in fact fiberwise diffeomorphic to

$$\frac{f}{\delta} : \{|f| = \delta\} \cap Y^* \cap \overset{\circ}{B} \rightarrow S^1. \quad (22)$$

The proof proceeds by construction of an outward pointing vector field on $Y^* \setminus Y_0$, whose flow takes $\{|f| = \delta\} \cap Y^* \cap B$ fiberwise onto $\Sigma^* \setminus \{|f| \leq \delta\}$. For two chosen single fibers we can then establish an isomorphism.

Unlike in the case of an ICIS it is not so easy to see that if we pass to the closure in B , we still get a fibration.

Lemma 2.12. *For sufficiently small $\delta > 0$ the map*

$$\frac{f}{\delta} : \{|f| = \delta\} \cap Y^* \cap B \rightarrow S^1 \quad (23)$$

is a C^∞ fiber bundle with closed fibers

$$F = \{f = \delta\} \cap Y^* \cap B.$$

¹There is a typo in [17]: The third case in the mentioned corollary is $2 < q \leq n - 1$.

Proof. By choice of the Milnor ball, there are no critical points of f on $(Y^* \setminus Y_0) \cap B$. Hence we only have to check that $f/|\delta|$ is a submersion at the boundary

$$\{|f| = \delta\} \cap \Sigma^*.$$

This can be achieved by using first the curve selection lemma to show that f has no critical points on $\Sigma^* \setminus \Sigma$ on a neighborhood U of Σ , and then the compactness of Σ : For sufficiently small δ the set $\{|f| \leq \delta\} \cap \Sigma^*$ will be contained in U . \square

To create the setup to prove theorem 2.11, we first choose $\alpha > 0$ such that

- all requirements of corollary 2.7 are fulfilled, so that we will have a fibration of the second boundary.
- the space

$$N_\alpha := \Sigma^* \cap \{q \leq \alpha^2\}$$

has $S = V \cap \Sigma$ as a strong deformation retract in Σ^* .

After that we choose $\delta > 0$ sufficiently small with respect to α such that

- again the assumptions of corollary 2.7 are met.
- lemma 2.12 holds and we get a Milnor fibration by f .
- we have Σ as a strong deformation retract of the space

$$\Sigma_{\leq \delta} := \Sigma^* \cap \{|f| \leq \delta\}$$

and the retraction takes the subset $\partial N_\alpha \cap \Sigma_{\leq \delta}$ into itself.

This last space now decomposes as

$$\Sigma_{\leq \delta} = (\Sigma_{\leq \delta} \cap \overline{(\Sigma^* \setminus N_\alpha)}) \cup (\Sigma_{\leq \delta} \cap N_\alpha) =: T_1 \cup T_2.$$

The attentive reader may recognize T_2 from corollary 2.7. The other part T_1 has a natural structure as a trivial disc bundle over Σ as in the case of isolated singularities, since $\Sigma \setminus N_\alpha$ was compact and smooth.

Now we can according to Hamm's computations decompose the space Σ^* as

$$\begin{aligned} \Sigma^* &\cong (\Sigma^* \cap \{|f| \leq \delta\}) \cup (\Sigma^* \cap \{|f| \geq \delta\}) \\ &\cong (\Sigma_{\leq \delta}) \cup (Y^* \cap B \cap f^{-1}(D_\delta)), \end{aligned}$$

where the second part is a smooth fiber bundle over the circle by lemma 2.12.

Proof. (of theorem 2.11) Consider the triple of spaces $(\Sigma^*, F \cup T_2, T_2)$. We have the following isomorphisms for the relative homology groups.

$$H_q(\Sigma^*, F \cup T_2) \cong H_q(\Sigma^*, F \cup T_2 \cup T_1) \tag{24}$$

$$\cong H_q(Y^* \cap B \cap \{|f| = \delta\}, F \cup (\Sigma^* \cap \{|f| = \delta\})) \tag{25}$$

$$\cong H_q(F \times [0, 1], \partial(F \times [0, 1])) \tag{26}$$

$$\cong H_{q-1}(F, \partial F) \otimes H_1(I, \partial I) = H_{q-1}(F, \partial F) \tag{27}$$

for $q > 0$ and $H_0(\Sigma^*, F \cup T_2) = 0$. The first line (24) holds, because T_1 retracts onto the part of the boundary of F outside N_α . By excision we get (25) and (26) comes from the fibration (lemma 2.12). Then (27) comes from the Künneth formula.

$$H_q(\Sigma^*, T_2) \cong H_q(\Sigma^*, N_\alpha \cap \Sigma) \quad (28)$$

$$\cong H_q(\Sigma^*, S), \quad (29)$$

because by assumption T_2 retracts onto $N_\alpha \cap \Sigma$ which in turn retracts onto $S = V \cap \Sigma$. Finally by excision we deduce

$$H_q(F \cup T_2, T_2) \cong H_q(F, \partial_2 F). \quad (30)$$

With these identifications the long exact sequence from the triple reads

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_{q+1}(\Sigma^*, F \cup T_2) & \longrightarrow & H_q(F \cup T_2, T_2) & \longrightarrow & H_q(\Sigma^*, T_2) \longrightarrow \cdots \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \cdots & \longrightarrow & H_q(F, \partial F) & \longrightarrow & H_q(F, \partial_2 F) & \longrightarrow & H_q(\Sigma^*, S) \longrightarrow \cdots \end{array} \quad (31)$$

Recall that according to Andreotti and Fraenkel's proof of the Lefschetz hyperplane theorem (see e.g. [12]), the relative homology groups $H_q(F, \partial F)$ vanish for $q < n$. Thus we find isomorphisms

$$H_q(F, \partial_2 F) \cong H_q(\Sigma^*, S) \quad \text{for } q < n. \quad (32)$$

To determine the connectivity of the pair $(F, \partial_2 F)$ we are therefore left with the computation of the relative homology groups $H_q(\Sigma^*, S)$.

The rest of the proof will split into three cases. In any of these, we will show that from the long exact sequence in homology of the pair (Σ^*, S) we get

$$H_q(\Sigma^*, S) \cong \begin{cases} 0 & \text{for } 2 < q < n \\ H_{q-1}(S) = \mathbb{Z} & \text{for } 0 < q \leq 2 \end{cases} \quad (33)$$

Case I: Y^* is smooth.

This has been done by Dirk Siersma in [17]. The pair (Σ^*, S) is just (S^{2n-1}, S^1) with the usual equatorial embedding. Clearly (33) holds and (20) follows for the case $0 \leq q < n, q \neq 2$. For $q = 2$ consider the following commutative diagram

$$\begin{array}{ccccccc} H_2(F, \partial_2 F) & \xrightarrow{\cong} & H_2(F \cup T_2, T_2) & \xrightarrow{\cong} & H_2(\Sigma^*, T_2) & \xrightarrow{\cong} & H_2(\Sigma^*, S) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \cong \\ H_1(\partial_2 F) & \xrightarrow{\cong} & H_1(T_2) & \xrightarrow{\cong} & H_1(T_2) & \xrightarrow{\cong} & H_1(S) \end{array} \quad (34)$$

All horizontal maps are isomorphisms. In the lower row they are induced by the inclusion $\partial_2 F \hookrightarrow T^2$ and the retraction of T_2 onto S . The vertical map on the right clearly is an isomorphism, too. This finishes the proof in case I.

Case II: Y^* has an isolated singular point at the origin.

In this case Σ^* is a smooth compact manifold. Let F^* be the Milnor fiber of the isolated complete intersection singularity $(Y^*, 0)$. The dimension of F^* is $n + 1$ and according to Hamm it is homotopic to a bouquet of $(n + 1)$ -dimensional spheres. The Lefschetz hyperplane theorem asserts that one can obtain F^* from Σ^* by attaching cells of dimension $\geq n + 1$. Then clearly Σ^* must be $(n - 1)$ -connected and (33) follows from the long exact sequence of the pair (Σ^*, S) . The proof is finished with the same arguments as in case I.

Case III: Y^* is also singular along V .

Here S the singular part of the boundary Σ^* of Y^* . For α sufficiently small, it is homotopic to the pair (Σ^*, N_α) . Let again F^* be the Milnor fiber in a smoothing of Y^* and consider the triple $(F^*, \partial F^*, \partial_2 F^*)$. By excision we clearly have isomorphisms

$$H_q(\Sigma^*, S) \cong H_q(\Sigma^*, N_\alpha) \cong H_q(\partial F^*, \partial_2 F^*)$$

for all q . The long exact sequence for the triple reads

$$\cdots \longrightarrow H_{q+1}(F^*, \partial F^*) \longrightarrow H_q(\partial F^*, \partial_2 F^*) \longrightarrow H_q(F^*, \partial_2 F^*) \longrightarrow \cdots$$

and for $q + 1 < n + 1 = \dim F^*$, the terms $H_{q+1}(F^*, \partial F^*)$ vanish. Thus for all $0 < q < n$ we have isomorphisms

$$H_q(F, \partial_2 F) \cong H_q(\Sigma^*, S) \cong H_q(\partial F^*, \partial_2 F^*) \cong H_q(F^*, \partial_2 F^*). \quad (35)$$

The claim now follows by induction on the codimension of Y^* . For $0 \leq q < n, q \neq 2$ the right hand term of (35) is zero and in case $q = 2$ we can extend the diagram (34) by one more column to obtain

$$\begin{array}{ccccccc} H_2(F, \partial_2 F) & \xrightarrow{\cong} & \cdots & \xrightarrow{\cong} & H_2(\Sigma^*, S) & \xrightarrow{\cong} & H_2(F^*, \partial_2 F^*) \\ \downarrow & & & & \downarrow & & \downarrow \cong \\ H_1(\partial_2 F) & \xrightarrow{\cong} & \cdots & \xrightarrow{\cong} & H_1(S) & \xrightarrow{\cong} & H_1(\partial_2 F^*) \end{array} \quad (36)$$

□

2.4.3 The surface case

We already saw in section 2.4.1, corollary 2.10, that surfaces need special treatment. The reason for this is that the horizontal and the vertical cycles of the second boundary $\partial_2 F$ do not live in distinct homology groups anymore. In view of its applications for ICMC2 singularities in the next section, we will formulate a different connectivity result for the pair $(F, \partial_2 F)$ in the case $n = 2$.

Theorem 2.13. *Let $(Y_0, 0) \subset (\mathbb{C}^N, 0)$ be a complete intersection line singularity of dimension $n = 2$. Recall the decomposition*

$$H_1(\partial_2 F) \cong H'_1 \oplus \mathbb{Z}$$

into horizontal and vertical cycles (19) for the second boundary $\partial_2 F$ of the Milnor fiber F of Y_0 . With these identifications, the natural map $\iota_1 : H_1(\partial_2 F) \rightarrow H_1(F)$ is surjective and factors via

$$\begin{array}{ccc} H'_1 \oplus \mathbb{Z} & \xrightarrow{\iota_1} & H_1(F) \\ & \searrow & \nearrow \\ & H'_1 & \end{array} \quad (37)$$

In other words: The vertical cycles are homologous to zero in F , while every remaining 1-cycle of F comes from a cycle in $\partial_2 F$.

Proof. We can literally copy the setup and the beginning of the proof of theorem 2.11 up to the point, where we deduce the isomorphisms (32). From here the proof of theorem 2.13 becomes an investigation of the part

$$H_2(F, \partial_2 F) \longrightarrow H_1(\partial_2 F) \longrightarrow H_1(F) \longrightarrow H_1(F, \partial_2 F) \longrightarrow 0$$

of the long exact sequence from the pair $(F, \partial_2 F)$.

Let $l : S^1 \rightarrow \partial_2 F$ be a section of $\arg L$ representing the homology class $[l]$ of the generator of the vertical cycles in $H_1(\partial_2 F)$. Consider the commutative diagram

$$\begin{array}{ccccccc} H_2(F, \partial_2 F) & \xrightarrow{\cong} & H_2(F \cup T_2, T_2) & \xrightarrow{\cong} & H_2(\Sigma^*, T_2) & \xrightarrow{\cong} & H_2(\Sigma^*, S) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \delta \\ H_1(\partial_2 F) & \xrightarrow{\kappa} & H_1(T_2) & \xrightarrow{\cong} & H_1(T_2) & \xrightarrow{\cong} & H_1(S) \end{array} \quad (38)$$

where the column maps are the natural ones from the corresponding pairs of spaces. Contrary to the higher dimensions, the map κ coming from the inclusion $\partial_2 F \hookrightarrow T_2$ is not necessarily an isomorphism anymore. But clearly it maps $[l]$ into the homology class of the generator of $H_1(S) \cong \mathbb{Z}$.

The factorization (37) would follow from δ on the right being surjective. Surjectivity of ι_1 in (37) directly follows from $H_1(F, \partial_2 F)$ being zero.

Case I: Y^* is smooth (cf. [17]).

The pair (Σ^*, S) is nothing but a pair of spheres (S^5, S^1) with the standard equatorial embedding. Clearly δ in (38) is surjective and from (32) we get

$$H_1(F, \partial_2 F) \cong H_1(S^5, S^1) = 0.$$

Case II: Y^* has an isolated singularity at the origin.

The Milnor fiber F^* of Y^* is a bouquet of spheres of dimension 3 and the pair

$(F^*, \partial F^*)$ is 2-connected. We consider a smoothing of $(Y^*, 0)$ compatible with the constructions made for $(Y_0, 0)$. The term

$$H_1(F, \partial_2 F) \cong H_1(\Sigma^*, S) \cong H_1(\partial F^*, \partial_2 F^*)$$

appears in the long exact sequence of the triple $(F^*, \partial F^*, \partial_2 F^*)$:

$$\cdots \longrightarrow H_2(F^*, \partial F^*) \longrightarrow H_1(\partial F^*, \partial_2 F^*) \longrightarrow H_1(F^*, \partial_2 F^*) \longrightarrow \cdots$$

Both terms on the left and the right are zero due to the connectivity of $(F^*, \partial F^*)$ and the long exact sequence of the pair $(F^*, \partial_2 F^*)$. Consequently $H_1(F, \partial_2 F)$ also is.

The space $\Sigma^* = \partial F^*$ is 1-connected, for if it wasn't, according to the connectivity with its boundary, F^* couldn't be a bouquet of spheres. This shows surjectivity of δ in (38).

Case III: Y^* is singular along V .

For the surjectivity of δ in (38), we apply the same argument as in the higher dimensional case. From the long exact sequence of the triple $(F^*, \partial F^*, \partial_2 F^*)$ and the connectivity of $(F^*, \partial F^*)$ and theorem 2.13 we deduce surjectivity of the natural map

$$H_2(\Sigma^*, S) \cong H_2(\partial F^*, \partial_2 F^*) \rightarrow H_2(F^*, \partial_2 F^*) \cong H_1(\partial_2 F).$$

With this we can extend the diagram (38) to the right by the column

$$\begin{array}{ccccccc} H_2(F, \partial_2 F) & \xrightarrow{\cong} & \cdots & \xrightarrow{\cong} & H_2(\Sigma^*, S) & \longrightarrow & H_2(F^*, \partial_2 F^*) \\ \downarrow & & & & \downarrow \delta & & \downarrow \cong \\ H_1(\partial_2 F) & \xrightarrow{\kappa} & \cdots & \xrightarrow{\cong} & H_1(S) & \xrightarrow{\cong} & H_1(\partial_2 F^*) \end{array} \quad (39)$$

Also from the connectivity of $(F^*, \partial_2 F^*)$ we get

$$H_1(F, \partial_2 F) \cong H_1(\Sigma^*, S) \cong H_1(F^*, \partial_2 F^*) = 0.$$

□

3 Application to ICMC2 singularities of type 2

Let $(X_0, 0) \subset (\mathbb{C}^{n+2}, 0)$ be an ICMC2 singularity of Cohen-Macaulay type $t = 2$ described by the matrix

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{pmatrix}.$$

so that the Tjurina transform $(Y_0, \{0\} \times \mathbb{P}^1) \subset (\mathbb{C}^{n+2} \times \mathbb{P}^1, \{0\} \times \mathbb{P}^1)$ is given by the equations

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} := \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{pmatrix} \cdot \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = 0.$$

Assume that Y_0 is singular along the exceptional set $V = \{0\} \times \mathbb{P}^1$. We may choose a Whitney stratification for Y_0 by strata

$$(Y_0 \setminus V, V \setminus \{p_1, \dots, p_N\}, \{p_1, \dots, p_N\}).$$

The first part of this section is devoted to creating a setup, in which the conditions for the methods and results of section 2 are met.

First we construct the space Y^* globally by the same arguments. Let $X_0 \subset U \subset \mathbb{C}^{n+2}$ be a representative of $(X_0, 0)$ in some open neighborhood U of the origin and $Y_0 \subset U \times \mathbb{P}^1$ its Tjurina transform. Consider

$$\mathbb{P}f : U \times \mathbb{P}^1 \setminus Y_0 \rightarrow \mathbb{P}^2, \quad (x, s) \mapsto (f_1(x, s) : f_2(x, s) : f_3(x, s)).$$

This is a well defined map although the f_i are not functions. Choose a regular value $z \in \mathbb{P}^2$ and define

$$Y^* := \overline{\mathbb{P}f^{-1}(\{z\})} \subset U \times \mathbb{P}^1.$$

After a change of coordinates of \mathbb{P}^2 sending z to $(0 : 0 : 1)$, which naturally translates to row operations on A , we can assume that Y^* is given by the equations $f_1 = f_2 = 0$ and that

$$Y_0 = \{f = 0\} \cap Y^* \subset Y^*$$

is the zero locus of $f := f_3 \in H^0(U \times \mathbb{P}^1, \mathcal{O}(1))$.

Next we define the polar curve. Let $L : Y^* \subset \mathbb{C}^{n+2} \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be the projection to \mathbb{P}^1 and $z \in \mathbb{P}^1$ a regular value of L on $Y_0 \setminus V$. We may, after a change of coordinates, which corresponds to a canonical column operation on A , assume that $z = (0 : 1) = \infty$. In the chart $\{s_1 \neq 0\}$ we can play the same game as in lemma 2.3 in the whole chart at once to obtain a bending of L , which is sufficiently general for our needs. Any chosen bending in this chart will not alter the fiber of L over ∞ .

Observe that on the overlap $\{s_1 \neq 0\} \cap \{s_2 \neq 0\}$ the polar loci of the functions f/s_1 and $f/s_2 = f/s_1 \cdot s_1/s_2$ with respect to L coincide. We can express L as s_2/s_1 . Then, because

$$d \frac{f}{s_1} = d \left(\frac{f}{s_2} \cdot \frac{s_2}{s_1} \right) = \frac{s_2}{s_1} \cdot d \frac{f}{s_2} + \frac{f}{s_2} \cdot d \frac{s_2}{s_1}$$

clearly

$$\begin{aligned}\Gamma &= \overline{\left\{x \in Y^* \setminus Y_0 : d \frac{f}{s_2}(x) \text{ and } d \frac{s_2}{s_1}(x) \text{ are linearly dependent in } \Omega_{Y^*}^1\right\}} \\ &= \overline{\left\{x \in Y^* \setminus Y_0 : d \frac{f}{s_1}(x) \text{ and } d \frac{s_2}{s_1}(x) \text{ are linearly dependent in } \Omega_{Y^*}^1\right\}}.\end{aligned}$$

After possibly repeating the bending process of L on the other chart, we have a well defined global polar curve $\Gamma \subset Y^*$, which is smooth outside Y_0 and meets Y_0 only at finitely many points along V . We add those points to the zero-dimensional stratum of the Whitney stratification of Y_0 .

3.1 The generic rank 1 perturbation and the axis

Because $f = a_{3,1} \cdot s_1 + a_{3,2} \cdot s_2$ is a section of $\mathcal{O}(1)$ and not a function on Y^* , we can not globally perturb by a constant, but we have to choose another section $b = b_0 \cdot s_1 + b_1 \cdot s_2 \in H^0(\mathbb{C}^{n+2} \times \mathbb{P}^1, \mathcal{O}(1))$ and consider

$$f - \delta \cdot b = 0$$

in $\mathbb{C}^{n+2} \times \mathbb{P}^1 \times \mathbb{C}$. Thus there will always be one point in V , the zero locus of b , at which we will perturb the local equation of f by zero. This point is called the *axis point* of the deformation. It is unavoidable, but we can choose its position by the parameters $(b_1 : b_2)$.

Let us assume that after a change of coordinates the point $(0, \infty) := (0, (0 : 1)) \in V$ is not in the stratum $\{p_1, \dots, p_N\}$ of Y_0 and consider the deformation, which has $(0, \infty)$ as the axis point. For the original ICMC2 singularity $(X_0, 0)$ this means we consider the deformation given by the perturbation

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{pmatrix} - \delta \cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (40)$$

This gives the equations for the total space $Y \subset \mathbb{C}^{n+2} \times \mathbb{P}^1 \times \mathbb{C}$ of the deformation of Y_0 in the obvious way.

Note, that due to the generality assumptions in the choices of Y^* and the axis point this is a *generic rank 1 perturbation*. Every perturbation of A by a constant matrix B of rank 1 can be brought to this form using row- and column operations on $A - \delta \cdot B$.

In the chart $\{s_1 \neq 0\}$ we now have a deformation of Y_0 given by the perturbation of

$$\frac{f}{s_1} : Y^* \setminus \{s_1 \neq 0\} \rightarrow \mathbb{C} \quad (41)$$

by δ . On the other hand at the axis point $(0, \infty)$ we find

$$\frac{f}{s_2} : (Y^*, (0, \infty)) \rightarrow (\mathbb{C}, 0) \quad (42)$$

perturbed by $\delta \cdot s$, where $s = s_1/s_2$ is the local coordinate of \mathbb{P}^1 at ∞ .

3.2 Y_δ at the axis point

By assumption the axis point $(0, \infty)$ of the generic rank 1 deformation was in general position along V . This means if we let $g = \frac{f}{s_2}$ be the local equation (42) for Y_0 in Y^* at the axis point, we find ourselves in the setup of theorem 2.5.

Let $s = s_1/s_2$ and x_1, \dots, x_{n+2} be local coordinates in this chart such that the point $(0, \infty)$ is the origin and choose $\alpha, \beta > 0$ as in theorem 2.5. Then for δ small enough the map

$$G := (g, s) : Y^* \cap \Delta_{\alpha\beta} \cap g^{-1}(D_\delta) \rightarrow D_\delta \times D_\beta$$

is a fiber bundle away from $Y_0 = G^{-1}(\{0\} \times D_\beta)$.

The Milnor fiber of g at 0 is the preimage of a line $\{\delta\} \times D_\beta$ under this map. It inherits its product structure from the fibration by s . To obtain the deformed fiber Y_δ of the generic rank 1 deformation of Y_0 at $(0, \infty)$ from , we have to take a bent line

$$W = \{(\delta \cdot y, y) : y \in D_\beta\} \subset D_\delta \times D_\beta.$$

Now $Y_\delta \cap \Delta_{\alpha\beta} = G^{-1}(W)$. We deduce the following lemma:

Lemma 3.1. *Let $g : Y^* \rightarrow \mathbb{C}$ be the local equation for Y_0 at the axis point $(0, \infty) \in V$, s a local coordinate for V at $(0, \infty)$ with $s(0, \infty) = 0$ and $\Delta_{\alpha\beta}$ a chosen polydisc in the sense of theorem 2.5. Then for $\delta > 0$ sufficiently small with respect to α and β , the space*

$$Y_\delta \cap \Delta_{\alpha\beta} := Y^* \cap \Delta_{\alpha\beta} \cap \{g = s \cdot \delta\}$$

is the fiber over δ of the generic rank 1 perturbation (42) close to the axis point. The map

$$L : Y_\delta \cap \Delta_{\alpha\beta} \rightarrow \mathbb{P}^1$$

is a fibration over a punctured neighborhood D_β^\times of $\infty \in \mathbb{P}^1$. The central fiber

$$Y_\infty^{\text{th}} := Y_\delta \cap \Delta_{\alpha\beta} \cap \{L = \infty\} = Y_0 \cap \Delta_{\alpha\beta} \cap \{L = \infty\},$$

however, does not change as we pass from Y_0 to Y_δ . Consequently Y_δ may retain a singular point at $(0, \infty)$. If this happens, it is at most an ICIS.

Definition 3.2. *The space Y_∞^{th} is called the axis of the deformation.*

Corollary 3.3. *The space $Y_\delta \cap \Delta_{\alpha\beta}$ as in lemma 3.1 is contractible.*

Proof. The central fiber Y_∞^{th} is a euclidean neighborhood retract of some open neighborhood U in $Y_\delta \cap \Delta_{\alpha\beta}$. Clearly the fiber bundle $(Y_\delta \cap \Delta_{\alpha\beta}) \setminus Y_\infty^{\text{th}}$ can be retracted onto U and successively onto Y_∞^{th} . Being the representative of a germ of an isolated singularity in a Milnor ball, Y_∞^{th} is contractible. Concatenation of these two contractions establishes the claim. \square

3.3 The global picture in the generic rank 1 perturbation

After we already have a description of what happens at the axis point $(0, \infty)$ in a generic rank 1 perturbation, let us now compute the topology of Y_δ in the other chart. To create a global setup first choose Milnor balls B_i of radius ε for all special points $\{p_1, \dots, p_N\}$ of Y_0 . Let B'_i be a Milnor ball of radius $\varepsilon/2$ around p_i and set

$$B = \bigcup_{i=1}^N B_i, \quad B' = \bigcup_{i=1}^N B'_i.$$

We now choose $\alpha > 0$ sufficiently small such that

- all the local theory (theorem 2.11 for threefolds or theorem 2.13 in the surface case) works at the special points p_i ,
- theorem 2.5 holds along the set

$$V' := V \setminus (B' \cup \{(0, \infty)\}),$$

i.e. for $\delta > 0$ small enough the map

$$L : Y_\delta \cap \{q \leq \alpha^2\} \cap L^{-1}(V') \rightarrow V' \quad (43)$$

is a fiber bundle over V' with fiber F^{th} .

Note that for the last requirement we can use lemma 3.1 to achieve this behaviour in a neighborhood of the axis point. After that we're left with a compact subset of V , along which the existence of a global minimal $\alpha > 0$ can certainly be assured.

Now we choose $\delta > 0$ small enough with respect to all prior choices such that all the local theory developed above works at once along all points of the compact set V .

We can now piece together the topology of Y_δ from the topology of the several known patches. We regard the axis point $(0, \infty)$ as a further special point p_0 in the Whitney stratification of Y_0 . Let Δ be the chosen polydisk around p_0 and set

$$U := Y_\delta \cap (B \cup \Delta), \quad W := Y_\delta \cap \{q \leq \alpha^2\} \cap L^{-1}(V'). \quad (44)$$

Furthermore let $\partial_2 F_i$ be the second boundary of the local Milnor fiber of (Y_0, p_i) at p_i for $i > 0$. In case $i = 0$, i.e. at the axis point, we just set

$$\partial_2 F_0 := Y_\delta \cap \{q \leq \alpha^2\} \cap L^{-1}(\partial D_\beta),$$

where D_β was the chosen disc around $\infty \in \mathbb{P}^1$. We easily verify that the inclusion

$$\partial_2 F_i \hookrightarrow (U \cap W)_i$$

induces a homology equivalence

$$H_q(U \cap W) \cong \bigoplus_{i=0}^N H_q((U \cap W)_i) \cong \bigoplus_{i=0}^N H_q(\partial_2 F_i) \quad (45)$$

where $(U \cap W)_i$ is the component of $U \cap W$ close to p_i . For $q = 1$ and $i > 0$ let $[l_i]$ be the generator of $H_1(\partial_2 F_i)$ – respectively the generator of the vertical part in case $n = 2$ – represented by a section $l_i : S^1 \rightarrow \partial_2 F_i$ of $\arg L$ in (16), cf. corollary 2.9 and 2.10.

The homology groups of W itself are determined by its structure as a fiber bundle over V' (43). Because V' has the homotopy type of a finite bouquet of circles around the points p_1, \dots, p_N , we can basically repeat the arguments leading to corollary 2.9 and 2.10. In particular we can assume

$$H_0(W) = 0, \quad H_1(W) = \begin{cases} \mathbb{Z}^N & \text{if } n = 3 \\ H'_1 \oplus \mathbb{Z}^N & \text{if } n = 2 \end{cases} \quad (46)$$

where H'_1 is the quotient of $H_1(Y_\delta^{\text{rh}})$ by the monodromies around all loops in the base V' and in both cases \mathbb{Z}^N is generated by the $[l_i]$. We can view the latter as sections of the generators of $H_1(V')$.

Theorem 3.4. *Let Y_δ be the fiber over $\delta \neq 0$ in the genereric rank 1 perturbation of the Tjurina transform $(Y_0, V) \subset (\mathbb{C}^{n+2} \times \mathbb{P}^1, \{0\} \times \mathbb{P}^1)$ of an ICMC2 threefold singularity $(X_0, 0) \subset (\mathbb{C}^{n+2}, 0)$ of dimension $n = 2$ or 3 and Cohen-Macaulay type $t = 2$. Let $L : Y_\delta \rightarrow \mathbb{P}^1$ be the projection to \mathbb{P}^1 and $G \subset H^\bullet(Y_\delta)$ the image of $L^* : H^\bullet(\mathbb{P}^1) \rightarrow H^\bullet(Y_\delta)$. Then Y_δ is simply connected and the homology of Y_δ splits into*

$$H_\bullet(Y_\delta) \cong G^\perp \oplus \mathbb{Z},$$

where $G^\perp = \{[\sigma] \in H_\bullet(Y_\delta) : g \cap [\sigma] = 0 \quad \forall g \in G\}$ are the horizontal cycles of Y_δ . The cap product with $L^*(H^2(\mathbb{P}^1))$ gives a perfect pairing of the vertical cycles $H_\bullet(Y_\delta)/G^\perp = \mathbb{Z}$ with $H^2(\mathbb{P}^1)$. If $n = 3$, then $H_2(X_\varepsilon) \cong \mathbb{Z}$ consists of the vertical cycles only.

Proof. Consider the Mayer-Vietoris sequence for Y_δ for the choice (44) of the two patches U and W . First of all, the tail gives a short exact sequence

$$0 \longrightarrow H_0(U \cap W) \longrightarrow H_0(U) \oplus H_0(W) \longrightarrow H_0(Y_\delta) \longrightarrow 0$$

and Y_δ is clearly connected. The first homology group $H_1(Y_\delta)$ appears in the exact sequence

$$H_1(U \cap W) \xrightarrow{\iota_1} H_1(U) \oplus H_1(W) \longrightarrow H_1(Y_\delta) \longrightarrow 0. \quad (47)$$

We proceed with the proof for the case $n = 3$. From theorem 2.11 we know that $H_1(U) = 0$. On the generators chosen above, the map ι_1 to the second summand is given by the matrix

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 1 & 0 & \cdots & 0 & 1 \end{pmatrix} \quad (48)$$

and therefore clearly surjective. Thus $H_1(Y_\delta) = 0$ and Y_δ is simply connected.

Proceeding along the Mayer-Vietoris sequence to the left, we see in (49) that $H_2(Y_\delta)$ must be nonzero, because clearly the kernel of (48) is free of rank 1.

$$H_2(U) \oplus H_2(W) \xrightarrow{\kappa_2} H_2(Y_\delta) \xrightarrow{\partial_2} H_1(U \cap W) \xrightarrow{\iota_1} H_1(U) \oplus H_1(W) \quad (49)$$

But κ_2 is in fact the zero map. To see this observe the following. Every homology class $[\sigma] \in H_2(U)$ can be represented as a sum of 2-cycles in the boundaries

$$\sigma = \sum_{i=1}^N \sigma_i, \quad [\sigma_i] \in H_2(\partial_2 F_i)$$

as a consequence of theorem 2.11. Corollary 2.9 then tells us that $[\sigma_i]$ even comes from a cycle in a transversal Milnor fiber $[\sigma_i] \in H_2(F_i^{\text{th}})$ close to p_i . The same holds for any $[\sigma] \in H_2(W)$ and any other chosen transversal Milnor fiber over a point in V' .

Mapping any $[\sigma] \in H_2(U) \oplus H_2(W)$ into $H_2(Y_\delta)$ makes it therefore homologous to a cycle in a transversal Milnor fiber arbitrary close to Y_∞^{th} , the fiber of L over the axis point. Here it collapses, because $Y_\delta \cap \Delta$ was contractible by corollary 3.3.

Consequently $H_2(Y_\delta) = \ker \iota_1$. We construct a generator for $H_2(Y_\delta)$ as follows. Over $V' \cup \{\infty\}$ there exists a continuous section

$$l : V' \rightarrow Y_\delta \cap L^{-1}(V')$$

of L , because L gives $Y_\delta \cap L^{-1}(V')$ the structure of a fiber bundle with 1-connected fiber Y_δ^{th} over a base, which is homotopic to a bouquet of 1-spheres. We can extend l over ∞ , because we only glue in a contractible fiber. Let $D = \overline{\mathbb{P}^1 \setminus (V' \cup \{\infty\})}$ be the closure of the complement of the domain of definition of l . Then the fundamental class of the image of l defines a unique relative cycle

$$[l] \in H_2(Y_\delta, L^{-1}(D)).$$

Consider the following commutative diagram

$$\begin{array}{ccccccc} H_2(L^{-1}(D)) & \longrightarrow & H_2(Y_\delta) & \xrightarrow{\pi} & H_2(Y_\delta, L^{-1}(D)) & \longrightarrow & H_1(L^{-1}(D)) \\ & & \downarrow L_* & & \downarrow L_* & & \\ & & H_2(\mathbb{P}^1) & \xrightarrow{\cong} & H_2(\mathbb{P}^1, D) & & \end{array} \quad (50)$$

The image of $[l]$ in $H_1(L^{-1}(D))$ is zero by theorem 2.11: At each special point p_i the component of the boundary of $[l]$ in the local Milnor fiber is homologous to the generator of $H_1(\partial_2 F) \cong H_2(F, \partial_2 F)$. On the other hand the map on the left into $H_2(Y_\delta)$ is the zero map by the previous arguments: All 2-cycles of the local Milnor fibers become homologous to zero in Y_δ . A generator $[\sigma]$ of $H_2(Y_\delta)$ is therefore given as a preimage of $[l]$ under π .

The map L_* on the right is an isomorphism and hence on the left L_* maps $[\sigma]$ to the fundamental class of \mathbb{P}^1 . This finishes the proof for the threefolds.

If $n = 2$, we need to modify the arguments above. First we show surjectivity of ι_1 in (47). Recall that we can split

$$H_1(U \cap W) \cong \bigoplus_{i=0}^N H_1(\partial_2 F_i) \cong \bigoplus_{i=0}^N (H'_1(\partial_2 F_i) \oplus \mathbb{Z})$$

into its horizontal and vertical part, where $H'_1(\partial_2 F_i)$ is the cokernel of $H_1(F^\natural)$ by the vertical monodromy at p_i .

We can restrict the first component of ι_1 mapping into $H_1(U) = \bigoplus_{i=1}^N H'_1(F_i)$ to the summand $\bigoplus_{i=1}^N H_1(\partial_2 F_i)$ and the second component of ι_1 mapping into $H_1(W)$ to $H'_1(\partial_2 F_0) \oplus \mathbb{Z}^{N+1}$. Both restrictions themselves are surjective by theorem 2.13 and (46) and hence also ι_1 is.

On the vertical cycles \mathbb{Z}^N of $H_1(U \cap W)$ the map ι_1 takes again the same form (48) and consequently we can choose a splitting

$$H_2(Y_\delta) = H'_2(Y_\delta) \oplus \mathbb{Z}$$

of the second homology group of Y_δ with the second summand mapping to the kernel of ι_1 on the vertical cycles. We can construct a generator $[\sigma]$ of the quotient $H_2(Y_\delta)/H'_2(Y_\delta) = \mathbb{Z}$ similar to the threefold case. Start with a continuous section

$$l : V' \cup \{\infty\} \rightarrow Y_\delta$$

of $L : Y_\delta \rightarrow \mathbb{P}^1$. For surfaces the relative homology class $[l] \in H_2(Y_\delta, L^{-1}(D))$ is not unique, but depends on the choice of l . Nevertheless, its preimage $[\sigma] \in H_2(Y_\delta)$ under the map π in (50) generates the quotient $H_2(Y_\delta)/H'_2(Y_\delta) = \mathbb{Z}$ and the composite map $L_* \circ \pi$ is an isomorphism when restricted to the second summand of the splitting $H_2(Y_\delta) = H'_2(Y_\delta) \oplus \mathbb{Z}$.

Hence again $[\sigma]$ is mapped to the fundamental class of \mathbb{P}^1 by L_* . All other cycles in $H_2(Y_\delta)$ can be represented sitting in the preimage of discs or paths in \mathbb{P}^1 and are therefore mapped to zero by L_* . This finishes the proof for $n = 2$. \square

We can now prove the main theorem of this paper.

Proof. (of theorem 1.11) Consider a deformation of $(X_0, 0)$ with two parameters (δ, ε) , where the first one δ is for a generic rank 1 perturbation and the second one ε is for a smoothing. For the Tjurina transform $Y_{\delta,0}$ over $X_{\delta,0}$, the fiber over $(\delta, 0)$ for $\delta \neq 0$ small enough, the homology groups are described by theorem 3.4. However according to lemma 3.1, there might still be an ICIS of Y_δ at the axis point.

In case $Y_{\delta,0}$ is smooth, its diffeomorphism type does not change as we pass to a smooth fiber $Y_{\delta,\varepsilon}$ for $\delta, \varepsilon \neq 0$. If it was not, its topology changes at most at the axis point $(0, \infty)$, where it is the smoothing of an ICIS.

This means that, in the notation above, the local Milnor fiber $Y_{\delta,\varepsilon} \cap \Delta$ of $(Y_{\delta,0}, (0, \infty))$ is 3-connected. Hence all 1- and 2-cycles appearing in the proof of theorem 3.4 close to Y_∞^{rh} (i.e. representable by cycles in $Y_{\delta,\varepsilon} \cap \Delta$) still become homologous to zero in $Y_{\delta,\varepsilon}$ and we can literally repeat all the arguments. The theorem then follows from the isomorphism $Y_{\delta,\varepsilon} \cong X_{\delta,\varepsilon}$. \square

In this article we focused on ICMC2 singularities $(X_0, 0)$ of Cohen-Macaulay type 2, i.e. isolated determinantal singularities of type $(3, 2, 2)$. The reason is, that in this case the worst one can get in the Tjurina transform (Y_0, V) are line singularities. We saw that cycles from (Y_0, V) get passed on to the Milnor fiber X_ε and are then sitting over the homology of \mathbb{P}^1 by means of the map L associated to the deformed matrix. Computations for explicit examples as e.g. [5], example 3.5, yield that these phenomena can also be observed for ICMC2 singularities defined by bigger matrices.

Consider as another example the threefold singularity $(X_0, 0) \subset (\mathbb{C}^5, 0)$ defined by a generic embedding

$$A : \mathbb{C}^5 \hookrightarrow \text{Mat}(5, 4; \mathbb{C})$$

of a 5-dimensional subspace into $\text{Mat}(5, 4; \mathbb{C})$. The Tjurina transform now decomposes as

$$Y_0 = \overline{X_0} \cup (\{0\} \times \mathbb{P}^3) \subset \mathbb{C}^5 \times \mathbb{P}^3,$$

where $\overline{X_0}$ is the strict transform of X_0 and $\{0\} \times \mathbb{P}^3$ is an additional component. The locus

$$S = \overline{X_0} \cap (\{0\} \times \mathbb{P}^3),$$

where they meet, is a smooth projective hypersurface of degree 5, so we encounter “plane singularities” in the Tjurina transform in the sense that the singular locus itself has dimension two!

Nevertheless the induced families in the Tjurina transform coming from deformations of $(X_0, 0)$ are flat. Experimental computations show that the fiber Y_δ over $\delta \neq 0$ for a generic rank 2 perturbation is already smooth and hence diffeomorphic to the Milnor fiber X_ε . The axis of such a deformation is a whole projective line $H \subset \mathbb{P}^3$ and the fiber $Y_0 \cap L^{-1}(H) = Y_\delta \cap L^{-1}(H)$ of L sitting over it. This means that also the fundamental class of $H \cong \mathbb{P}^1$ is passed on to X_ε and then sitting over the corresponding cycle in \mathbb{P}^3 . Yet to develop a complete description of the topology of X_ε in the spirit of this paper, we would need to deal with singular loci of dimension 2 and their interplay with the topology of S and the axis – a task, which is far more evolved than what has been done in this article.

Apparently we gave a description in a special case of a more general phenomenon which yet remains to be explored: The vertical and horizontal vanishing cycles of isolated determinantal singularities.

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